

SEMISTAR INVERTIBILITY ON INTEGRAL DOMAINS

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ABSTRACT. After the introduction in 1994, by Okabe and Matsuda, of the notion of semistar operation, many authors have investigated different aspects of this general and powerful concept. A natural development of the recent work in this area leads to investigate the concept of invertibility in the semistar setting. In this paper, we will show the existence of a “theoretical obstruction” for extending many results, proved for star-invertibility, to the semistar case. For this reason, we will introduce two distinct notions of invertibility in the semistar setting (called \star -invertibility and quasi- \star -invertibility), we will discuss the motivations of these “two levels” of invertibility and we will extend, accordingly, many classical results proved for the d -, v -, t - and w -invertibility. Among the main properties proved here, we mention the following: **(a)** several characterizations of \star -invertibility and quasi- \star -invertibility and necessary and sufficient conditions for the equivalence of these two notions; **(b)** the relations between the \star -invertibility (or quasi- \star -invertibility) and the invertibility (or quasi-invertibility) with respect to the semistar operation of finite type (denoted by \star_f) and to the stable semistar operation of finite type (denoted by $\bar{\star}$), canonically associated to \star ; **(c)** a characterization of the $H(\star)$ -domains in terms of semistar-invertibility (note that the $H(\star)$ -domains generalize, in the semistar setting, the H -domains introduced by Glaz and Vasconcelos); **(d)** for a semistar operation of finite type a nonzero finitely generated (fractional) ideal I is \star -invertible (or, equivalently, quasi- \star -invertible, in the stable semistar case) if and only if its extension to the Nagata semistar ring $INa(D, \star)$ is an invertible ideal of $Na(D, \star)$.

1. INTRODUCTION AND BACKGROUND RESULTS

The notions of t -invertibility, v -invertibility and w -invertibility, that generalize the classical concept of (d) -invertibility (these definitions will be recalled in Section 2), have been introduced in the recent years for a better understanding of the multiplicative (ideal) structure of integral domains. In particular, t -invertibility has a key role for extending the notion of class group from Krull domains to general integral domains (cf. [8], [9], [10] and the survey paper [7]). An interesting chart of a large set of various t -, v -, d -invertibility based characterizations of several classes of integral domains can be found at the end of [4]; some motivations for introducing the w -invertibility and the first properties showing the “good” behaviour of this notion can be found in [47] (cf. also [30]). The concept of star operation (or, equivalently, ideal system, cf. the books by Jaffard [33], Gilmer [24] and Halter-Koch [27]) provides an abstract setting for approaching these different aspects of invertibility. A recent paper by Zafrullah [51] gives an excellent and updated survey of this point of view.

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After the introduction in 1994, by Okabe and Matsuda [44], of the notion of semistar operation, as a more general and natural setting for studying multiplicative systems of ideals and modules, many authors have investigated the possible extensions to the semistar setting of different aspects of the classical theory of ideal systems, based on the pioneering work by W. Krull, E. Noether, H. Prüfer and P. Lorenzen from 1930's (cf. for instance [40], [39], [45], [37], [38], [13], [28], [17], [18], [19], [12], [16], [15], [29], [41], [42], [43]).

A natural development of this work leads to investigate the concept of invertibility in the semistar setting. This is the purpose of the present paper, in which we will show the existence of a “theoretical obstruction” for extending many results, proved for star-invertibility, to the semistar case. For this reason, we will be forced to introduce two distinct notions of invertibility in the semistar setting (called \star -invertibility and quasi- \star -invertibility; the explicit definitions will be given in Section 2), we will discuss the motivations of these “two levels” of invertibility and we will extend, accordingly, many classical results proved for the d -, v -, t - and w -invertibility.

Among the main properties proved in this work, we mention the following: **(a)** several characterizations of \star -invertibility and quasi- \star -invertibility and necessary and sufficient conditions for the equivalence of these two notions; **(b)** the relations between the \star -invertibility (or quasi- \star -invertibility) and the invertibility (or quasi-invertibility) with respect to the semistar operation of finite type (denoted by \star_f) and to the stable semistar operation of finite type (denoted by $\widetilde{\star}$), canonically associated to \star [in case, $\star = v$ is the Artin's v -operation, then $\star_f = t$ and $\widetilde{\star} = w$]; **(c)** a characterization of the $H(\star)$ -domains in terms of semistar-invertibility (note that the $H(\star)$ -domains generalize in the semistar setting the H -domains introduced by Glaz and Vasconcelos [26], more precisely, we will see in Section 2 that an H -domain coincides with an $H(v)$ -domain); **(d)** for a semistar operation of finite type a nonzero finitely generated (fractional) ideal I is \star -invertible (or, equivalently, quasi- \star -invertible, in the stable semistar case) if and only if its extension to the Nagata semistar ring $INa(D, \star)$ is an invertible ideal of $Na(D, \star)$ (the definition of $Na(D, \star)$ will be recalled at the end of this section).

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Let D be an integral domain with quotient field K . Let $\overline{\mathbf{F}}(D)$ denote the set of all nonzero D -submodules of K and let $\mathbf{F}(D)$ be the set of all nonzero fractional ideals of D , i.e. $E \in \mathbf{F}(D)$ if $E \in \overline{\mathbf{F}}(D)$ and there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\mathbf{f}(D)$ be the set of all nonzero finitely generated D -submodules of K . Then, obviously $\mathbf{f}(D) \subseteq \mathbf{F}(D) \subseteq \overline{\mathbf{F}}(D)$.

A map $\star : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$, $E \mapsto E^\star$, is called a *semistar operation* on D if, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\mathbf{F}}(D)$, the following properties hold:

- (\star_1) $(xE)^\star = xE^\star$;
- (\star_2) $E \subseteq F$ implies $E^\star \subseteq F^\star$;
- (\star_3) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

cf. for instance [13]. Recall that [13, Theorem 1.2 and p. 174], for all $E, F \in \overline{\mathbf{F}}(D)$, we have :

$$\begin{aligned} (EF)^\star &= (E^\star F)^\star = (EF^\star)^\star = (E^\star F^\star)^\star ; \\ (E + F)^\star &= (E^\star + F)^\star = (E + F^\star)^\star = (E^\star + F^\star)^\star ; \\ (E : F)^\star &\subseteq (E^\star : F^\star) = (E^\star : F) = (E^\star : F)^\star ; \\ (E \cap F)^\star &\subseteq E^\star \cap F^\star = (E^\star \cap F^\star)^\star , \quad \text{if } E \cap F \neq (0) ; \end{aligned}$$

When $D^\star = D$, we say that \star is a *(semi)star operation*, since, restricted to $\mathbf{F}(D)$ it is a star operation.

For star operations it is very useful the notion of \star -ideal, that is, a nonzero ideal $I \subseteq D$, such that $I^\star = I$. For semistar operations we need a more general notion, that coincides with the notion of \star -ideal, when \star is a (semi)star operation. We say that a nonzero (integral) ideal I of D is a *quasi- \star -ideal* if $I^\star \cap D = I$. For example, it is easy to see that, if $I^\star \neq D^\star$, then $I^\star \cap D$ is a quasi- \star -ideal that contains I (in particular, a \star -ideal is a quasi- \star -ideal). Note that $I^\star \neq D^\star$ is equivalent to $I^\star \cap D \neq D$.

A *quasi- \star -prime* is a quasi- \star -ideal which is also a prime ideal. We call a *quasi- \star -maximal* a maximal element in the set of all proper quasi- \star -ideals of D . We denote by $\text{QSpec}^\star(D)$ (respectively, $\text{QMax}^\star(D)$) the set of all quasi- \star -prime (respectively, quasi- \star -maximal).

If \star is a semistar operation on D , then we can consider a map $\star_f : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$ defined for each $E \in \overline{\mathbf{F}}(D)$ as follows: $E^{\star_f} := \bigcup \{F^\star \mid F \in \mathbf{f}(D) \text{ and } F \subseteq E\}$. It is easy to see that \star_f is a semistar operation on D , called *the semistar operation of finite type associated to \star* . Note that, for each $F \in \mathbf{f}(D)$, $F^\star = F^{\star_f}$. A semistar operation \star is called a *semistar operation of finite type* if $\star = \star_f$. It is easy to see that $(\star_f)_f = \star_f$ (that is, \star_f is of finite type).

If \star_1 and \star_2 are two semistar operations on D , we say that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$, for each $E \in \overline{\mathbf{F}}(D)$. In this situation, it is easy to see that $(E^{\star_1})^{\star_2} = E^{\star_2} = (E^{\star_2})^{\star_1}$. Obviously, for each semistar operation \star , we have $\star_f \leq \star$.

The following result, with a different terminology, was proved in [13] (cf. also [19, Lemma 2.3]).

Lemma 1.1. *Let \star be a semistar operation on an integral domain D . Assume that \star is non trivial and $\star = \star_f$. Then:*

- (1) *Each proper quasi- \star -ideal is contained in a quasi- \star -maximal.*
- (2) *Each quasi- \star -maximal is a quasi- \star -prime.*
- (3) *Set $\Pi^\star := \{P \in \text{Spec}(D) \mid P \neq 0 \text{ and } P^\star \cap D \neq D\}$, then $\text{QSpec}^\star(D) \subseteq \Pi^\star$ and the set of maximal elements Π_{\max}^\star of Π^\star is nonempty and coincides with $\text{QMax}^\star(D)$.* \square

For the sake of simplicity, we will denote simply by $\mathcal{M}(\star)$ the set $\text{QMax}^\star(D)$ of the quasi- \star -maximal ideals of D .

If $\Delta \subseteq \text{Spec}(D)$, the map $\star_\Delta : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$, $E \mapsto E^{\star_\Delta} := \bigcap \{ED_P \mid P \in \Delta\}$, is a semistar operation. If $\star = \star_\Delta$, for some $\Delta \subseteq \text{Spec}(D)$, we say that \star is a *spectral semistar operation*. In particular, if $\Delta = \{P\}$, then $\star_{\{P\}}$ is the semistar operation on D defined by $E^{\star_{\{P\}}} := ED_P$, for each $E \in \overline{\mathbf{F}}(D)$. We say that a *semistar operation is stable* if $(E \cap F)^\star = E^\star \cap F^\star$, for each $E, F \in \overline{\mathbf{F}}(D)$. A spectral semistar operation is stable [13, Lemma 4.1].

If \star is a semistar operation on D , we denote by $\tilde{\star}$ the semistar operation $\star_{\mathcal{M}(\star_f)}$ induced by the set $\mathcal{M}(\star_f)$ of the quasi- \star_f -maximal ideals of D . The semistar operation $\tilde{\star}$ is stable and of finite type and $\tilde{\star} \leq \star_f$ (cf. [13, p. 181], where the semistar operation $\tilde{\star}$ is defined, in an equivalent way, by using localizing systems, and also [3, Section 2] for an analogous construction in the star setting). Note that when $\star = v$ (where, as usual, v denotes the (semi)star operation defined by $E^v := (D : (D : E))$, for each $E \in \overline{\mathbf{F}}(D)$), then $\tilde{\star}$ coincides with the (semi)star operation denoted by w by Wang Fanggui and R.L. McCasland (cf. [47], [48] and [49]).

The following lemma is not difficult to prove (cf. [19, Corollary 3.5(2)] and, for the analogous result in case of star operations, [3, Theorem 2.16]).

Lemma 1.2. *Let \star be a semistar operation on an integral domain D . Then, $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$.* \square

In the next proposition, we recall how a semistar operation on an integral domain D induces canonically a semistar operation on an overring T of D (cf. [44, Lemma 45], and, for the notation used here, [20]).

Proposition 1.3. *Let D be an integral domain and T an overring of D . Let $\iota : D \hookrightarrow T$ be the embedding of D in T , and let $\star_\iota : \overline{\mathbf{F}}(T) \rightarrow \overline{\mathbf{F}}(T)$ be defined by $E^{\star_\iota} := E^\star$. Then:*

- (1) \star_ι is a semistar operation on T .
- (2) If \star is of finite type on D , then \star_ι is of finite type on T .
- (3) If $T = D^\star$, then \star_ι is a (semi)star operation on D^\star .
- (4) If \star is stable, then \star_ι is stable. \square

If R is a ring and X an indeterminate over R , then the ring $R(X) := \{f/g \mid f, g \in R[X] \text{ and } \mathbf{c}(g) = R\}$ (where $\mathbf{c}(g)$ is the content of the polynomial g) is called the *Nagata ring* of R [24, Proposition 33.1].

The following result is proved in [19, Proposition 3.1] (cf. also [34, Proposition 2.1]).

Proposition 1.4. *Let \star be a nontrivial semistar operation on an integral domain D and set $N(\star) := N_D(\star) := \{h \in D[X] \mid h \neq 0 \text{ and } (\mathbf{c}(h))^\star = D^\star\}$. Then:*

- (1) $N(\star)$ is a saturated multiplicative subset of $D[X]$ and $N(\star) = N(\star_f) = D[X] \setminus \bigcup \{Q[X] \mid Q \in \mathcal{M}(\star_f)\}$.
- (2) $\text{Max}(D[X]_{N(\star)}) = \{Q[X]_{N(\star)} \mid Q \in \mathcal{M}(\star_f)\}$ and $\mathcal{M}(\star_f)$ coincides with the canonical image in $\text{Spec}(D)$ of $\text{Max}((D[X])_{N(\star)})$.
- (3) $D[X]_{N(\star)} = \bigcap \{D_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$. \square

We set $\text{Na}(D, \star) := D[X]_{N(\star)}$ and we call it the *Nagata ring of D with respect to the semistar operation \star* . Obviously, $\text{Na}(D, \star) = \text{Na}(D, \star_f)$ and, when $\star = d$ (the identity (semi)star operation) on D , then $\text{Na}(D, d) = D(X)$.

2. SEMISTAR INVERTIBILITY

Let \star be a semistar operation on an integral domain D . Let $I \in \mathbf{F}(D)$, we say that I is \star -invertible if $(II^{-1})^\star = D^\star$. In particular when $\star = d$ [respectively, v , t ($:= v_f$), w ($:= \tilde{v}$)] is the identity (semi)star operation [respectively, the v -operation, the t -operation, the w -operation] we reobtain the classical notion of *invertibility* [respectively, v -invertibility, t -invertibility, w -invertibility] of a fractional ideal.

Lemma 2.1. *Let \star, \star_1, \star_2 be semistar operations on an integral domain D . Let $\text{Inv}(D, \star)$ be the set of all \star -invertible fractional ideals of D and $\text{Inv}(D)$ (instead of $\text{Inv}(D, d)$) the set of all invertible fractional ideals of D . Then:*

- (0) $D \in \text{Inv}(D, \star)$.
- (1) If $\star_1 \leq \star_2$, then $\text{Inv}(D, \star_1) \subseteq \text{Inv}(D, \star_2)$. In particular, $\text{Inv}(D) \subseteq \text{Inv}(D, \tilde{\star}) \subseteq \text{Inv}(D, \star_f) \subseteq \text{Inv}(D, \star)$.
- (2) $I, J \in \text{Inv}(D, \star)$ if and only if $IJ \in \text{Inv}(D, \star)$.
- (3) If $I \in \text{Inv}(D, \star)$ then $I^{-1} \in \text{Inv}(D, \star)$.
- (4) If $I \in \text{Inv}(D, \star)$ then $I^v \in \text{Inv}(D, \star)$.

Proof. (0) and (1) are obvious.

(2) Note that, if $I, J \in \text{Inv}(D, \star)$, then $D^\star = (II^{-1})^\star (JJ^{-1})^\star \subseteq (II^{-1}JJ^{-1})^\star \subseteq (IJ(IJ)^{-1})^\star \subseteq D^\star$. Thus, $IJ \in \text{Inv}(D, \star)$. Conversely, if $IJ \in \text{Inv}(D, \star)$, then $D^\star = ((IJ)(D : IJ))^\star = (I(J(D : IJ)))^\star$. Since $(J(D : IJ)) \subseteq (D : I)$, it follows $(I(D : I))^\star = D^\star$. Similarly, $(J(D : J))^\star = D^\star$.

(3) $D^\star = (II^{-1})^\star \subseteq ((I^{-1})^{-1}I^{-1})^\star \subseteq D^\star$.

(4) follows from (3). \square

Remark 2.2. (a) Note that D is the unit element of $\text{Inv}(D, \star)$ with respect to the *standard multiplication* of fractional ideals of D . Nevertheless, $\text{Inv}(D, \star)$ is *not* a group in general (under the standard multiplication), because for $I \in \text{Inv}(D, \star)$, then $I^{-1} \in \text{Inv}(D, \star)$, but $II^{-1} \neq D$, if $I \notin \text{Inv}(D)$. For instance, let k be a field, X and Y two indeterminates over k , and let $D := k[X, Y]_{(X, Y)}$. Then D is a local Krull domain, with maximal ideal $M := (X, Y)D$. Let $\star = v$, then clearly $M^v = D$, since $\text{ht}(M) = 2$, thus M is v -invertible but M is not invertible in D , since it is not principal. Therefore $(MM^{-1})^v = D$, but $M = MM^{-1} \subsetneq D$. We will discuss later what happens if we consider the semistar (fractional) ideals semistar invertible with the “semistar product”.

(b) Let $I \in \mathbf{F}(D)$. Assume that $I \in \text{Inv}(D, \star)$ and $(D^\star : I) \in \mathbf{F}(D)$, then we will see later that $(D^\star : I) = (D : I)^\star$ (Lemma 2.10, Remark 2.13(d1) and Proposition 2.16), more precisely that:

$$(I^{-1})^\star = (D : I^v)^\star = (D^\star : I)^\star = (D^\star : I) = (I^\star)^{-1}.$$

However, in this situation, we may not conclude that $(D^\star : I)$ (or, $(D : I)^\star$) belongs to $\text{Inv}(D, \star)$ (even if $(D : I) \in \text{Inv}(D, \star)$, by Lemma 2.1(3)). As a matter of fact, more generally, if $J \in \text{Inv}(D, \star)$ and $J^\star \in \mathbf{F}(D)$, then J^\star does not belong necessarily to $\text{Inv}(D, \star)$.

For instance, let K be a field and X, Y two indeterminates over K , set $T := K[X, Y]$ and $D := K + YK[X, Y]$. Let $\star_{\{T\}}$ be the semistar operation on D defined by $E^{\star_{\{T\}}} := ET$, for each $E \in \overline{\mathbf{F}}(D)$. Then $J := YD$ is obviously invertible (hence $\star_{\{T\}}$ -invertible) in D and $J^{\star_{\{T\}}} = JT = YT = YK[X, Y] = (D : T)$ is a nonzero (maximal) ideal of D (and, at the same time, a (prime) ideal of T), but $J^{\star_{\{T\}}}$ is not $\star_{\{T\}}$ -invertible in D , because $(J^{\star_{\{T\}}}(D : J^{\star_{\{T\}}}))^{\star_{\{T\}}} = (JT(D : JT))T = (YT(D : YT))T = (YTY^{-1}(D : T))T = (T(YT))T = YT \subsetneq T = D^{\star_{\{T\}}}$.

(c) Note that the converses of (3) and (4) of Lemma 2.1 are not true in general. For instance, take an integral domain D that is not an H-domain (recall that an *H-domain* is an integral domain D such that, if I is an ideal of D with $I^{-1} = D$, then there exists a finitely generated $J \subseteq I$, such that $J^{-1} = D$ [26, Section 3]). Then, there exists an ideal I of D such that $I^v = I^{-1} = D$ and $I^t \subsetneq D$. It follows that $(I^{-1}I^v)^t = D$ (and so, I^{-1} and I^v are t -invertibles), but $(II^{-1})^t = I^t \subsetneq D$, that is, I is not t -invertible. On the other hand, note that, trivially, I is v -invertible.

An explicit example is given by a 1-dimensional non discrete valuation domain V with maximal ideal M . Clearly, V is not an H-domain [26, (3.2d)], $M^{-1} = M^v = V$ [24, Exercise 12 p.431] and $M^t = \bigcup \{J^v \mid J \subseteq M, J \text{ finitely generated}\} = \bigcup \{J \mid J \subseteq M, J \text{ finitely generated}\} = M \subsetneq V$. In this case, M^{-1} and M^v are obviously t -invertibles, but M is not t -invertible.

If $I \in \overline{\mathbf{F}}(D)$, we say that I is \star -finite if there exists $J \in \mathbf{f}(D)$ such that $J^\star = I^\star$. It is immediate to see that if $\star_1 \leq \star_2$ are semistar operations and I is \star_1 -finite, then I is \star_2 -finite. In particular, if I is \star_f -finite, then it is \star -finite.

We notice that, in the previous definition of \star -finite, we do not require that $J \subseteq I$. Next result shows that, with this “extra” assumption, \star -finite is equivalent to \star_f -finite.

Lemma 2.3. *Let \star be a semistar operation on an integral domain D with quotient field K . Let $I \in \overline{\mathbf{F}}(D)$. Then, the following are equivalent:*

- (i) I is \star_f -finite.
- (ii) There exists $J \subseteq I$, $J \in \mathbf{f}(D)$ such that $J^\star = I^\star$.

Proof. It is clear that (ii) implies (i), since $J^\star = J^{\star_f}$, if J is finitely generated. On the other hand, suppose I \star_f -finite. Then, $I^{\star_f} = J_0^{\star_f}$, with $J_0 = (a_1, a_2, \dots, a_n)D$,

for some family $\{a_1, a_2, \dots, a_n\} \subseteq K$. Since $J_0 \subseteq I^{\star_f}$, there exists a finite family of finitely generated fractional ideals of D , $J_1, J_2, \dots, J_n \subseteq I$, such that $a_i \in J_i^{\star}$, for $i = 1, 2, \dots, n$. It follows that $I^{\star_f} = J_0^{\star_f} \subseteq (J_1^{\star_f} + J_2^{\star_f} + \dots + J_n^{\star_f})^{\star_f} = (J_1 + J_2 + \dots + J_n)^{\star_f} \subseteq I^{\star_f}$. Set $J := J_1 + J_2 + \dots + J_n$. Then, J is finitely generated, $J \subseteq I$ and $J^{\star_f} = I^{\star_f}$, thus $J^{\star} = I^{\star}$. \square

Remark 2.4. Extending the terminology introduced by Zafrullah in the star setting [50] (cf. also [51, p. 433]), given a semistar operation on an integral domain D , we can say that $I \in \overline{\mathbf{F}}(D)$ is *strictly \star -finite* if $I^{\star} = J^{\star}$, for some $J \in \mathbf{f}(D)$, with $J \subseteq I$. With this terminology, Lemma 2.3 shows that \star_f -finite coincides with strictly \star_f -finite. This result was already proved, in the star setting, by Zafrullah [50, Theorem 1.1]. Note that Querré studied the strictly v -finite ideals [46], using often the terminology of *quasi-finite ideals*.

For examples of \star -finite ideals that are not \star_f -finite (when \star is the v -operation), see [22, Section (4c)], where are described domains with all the ideals v -finite (called DVF-domains), that are not Mori domains (that is, such that not all the ideals are t -finite).

Lemma 2.5. *Let \star be a semistar operation on an integral domain D and let $I \in \mathbf{F}(D)$. Then I is \star_f -invertible if, and only if, $(I'I'')^{\star} = D^{\star}$, for some $I' \subseteq I, I'' \subseteq I^{-1}$, and $I', I'' \in \mathbf{f}(D)$. Moreover, $I^{\star} = I^{\star}$ and $I''^{\star} = (I^{-1})^{\star}$.*

Proof. The “if” part is trivial. For the “only if”: if $(II^{-1})^{\star_f} = D^{\star_f}$, then $H^{\star} = D^{\star}$ for some $H \subseteq II^{-1}$, $H \in \mathbf{f}(D)$. Therefore, $H = (h_1, h_2, \dots, h_n)D$, with $h_i = x_{1,i}y_{1,i} + x_{2,i}y_{2,i} + \dots + x_{k_i,i}y_{k_i,i}$, with the x ’s in I and the y ’s in I^{-1} . Let I' be the (fractional) ideal of D generated by the x ’s and let I'' be the (fractional) ideal of D generated by the y ’s. Then, $H \subseteq I'I'' \subseteq II^{-1}$ and so $D^{\star} = (I'I'')^{\star}$, and, thus, also $D^{\star} = (I'I^{-1})^{\star} = (II'')^{\star}$. Moreover, $I^{\star} = (ID^{\star})^{\star} = (I(I'I^{-1})^{\star})^{\star} = ((II^{-1})^{\star}I')^{\star} = (D^{\star}I')^{\star} = I^{\star}$. In a similar way, we obtain also that $I''^{\star} = (I^{-1})^{\star}$. \square

A classical result due to Krull [33, Théorème 8, Ch. I, §4] shows that, for a star operation of finite type, star-invertibility implies star-finiteness. The following result gives a more complete picture of the situation in the general semistar setting.

Proposition 2.6. *Let \star be a semistar operation on an integral domain D . Let $I \in \mathbf{F}(D)$. Then I is \star_f -invertible if and only if I and I^{-1} are \star_f -finite (hence, in particular, \star -finite) and I is \star -invertible.*

Proof. The “only if” part follows from Lemma 2.5 and from the fact that $\star_f \leq \star$.

For the “if” part, note that by assumption $I^{\star_f} = J'^{\star_f} = J'^{\star}$ and $(I^{-1})^{\star_f} = J''^{\star_f} = J''^{\star}$, with $J', J'' \in \mathbf{f}(D)$. Therefore:

$$(II^{-1})^{\star_f} = (J'J'')^{\star_f} = (J'J'')^{\star} = (J'^{\star}J''^{\star})^{\star} = (I^{\star}(I^{-1})^{\star})^{\star} = (II^{-1})^{\star} = D^{\star}.$$

\square

Next goal is to investigate when the \star -invertibility coincides with the \star_f -invertibility.

Let \star be a semistar operation on an integral domain D , we say that D is an $H(\star)$ -domain if, for each nonzero integral ideal I of D such that $I^{\star} = D^{\star}$, there exists $J \in \mathbf{f}(D)$ with $J \subseteq I$ and $J^{\star} = D^{\star}$. It is easy to see that, for $\star = v$, the $H(v)$ -domains coincide with the H -domains introduced by Glaz and Vasconcelos (Remark 2.2(c)).

Lemma 2.7. *Let \star be a semistar operation on an integral domain D . Then D is an $H(\star)$ -domain if and only if each quasi- \star_f -maximal ideal of D is a quasi- \star -ideal of D .*

Proof. Assume that D is an $H(\star)$ -domain. Let $Q = Q^{\star_f} \cap D$ be a quasi- \star_f -maximal ideal of D . Assume that $Q^{\star} = D^{\star}$. Then, for some $J \in \mathbf{f}(D)$, with $J \subseteq Q$, we have $J^{\star} = D^{\star}$, thus $Q^{\star_f} = D^{\star}$, which leads to a contradiction. Therefore $Q^{\star_f} \cap D \subseteq Q^{\star} \cap D \subsetneq D$ and, hence, there exists a quasi- \star_f -maximal ideal of D containing $Q^{\star} \cap D$. This is possible only if $Q^{\star_f} \cap D = Q^{\star} \cap D$.

Conversely, let I be a nonzero ideal of D with the property $I^{\star} = D^{\star}$. Then, necessarily $I \not\subseteq Q$ for each quasi- \star_f -maximal ideal of D (because, otherwise, by assumption $I \subseteq Q = Q^{\star_f} \cap D = Q^{\star} \cap D$, and so $I^{\star} \subseteq Q^{\star} \subsetneq D^{\star}$). Therefore $I^{\star_f} = D^{\star}$. \square

Next result provides several characterizations of the $H(\star)$ -domains; note that, in the particular case that $\star = v$, the equivalence (i) \Leftrightarrow (iii) was already known [31, Proposition 2.4] and the equivalence (i) \Leftrightarrow (iv) was considered in [47, Proposition 5.7].

Proposition 2.8. *Let \star be a semistar operation on an integral domain D . The following are equivalent:*

- (i) D is an $H(\star)$ -domain;
- (ii) for each $I \in \mathbf{F}(D)$, I is \star -invertible if and only if I is \star_f -invertible;
- (iii) $\mathcal{M}(\star_f) = \mathcal{M}(\star)$;
- (iv) $\mathcal{M}(\widetilde{\star}) = \mathcal{M}(\star)$.

Proof. Obviously, (iii) \Leftrightarrow (iv) by Lemma 1.2 and (iii) \Leftrightarrow (i) by Lemma 2.7, recalling that a quasi- \star -ideal is also a quasi- \star_f -ideal.

(iii) \Rightarrow (ii). Let I be a \star -invertible ideal of D . Assume that I is not \star_f -invertible. Then, there exists a quasi- \star_f -maximal ideal M such that $II^{-1} \subseteq M$. But M is also quasi- \star -maximal, since $\mathcal{M}(\star_f) = \mathcal{M}(\star)$. Thus $M^{\star} \subsetneq D^{\star}$. It follows that $(II^{-1})^{\star} \subseteq M^{\star} \subsetneq D^{\star}$, a contradiction. Hence I is \star_f -invertible.

(ii) \Rightarrow (i) Let I be a nonzero integral ideal I of D such that $I^{\star} = D^{\star}$. Then, $I \subseteq II^{-1} \subseteq D$ implies that $(II^{-1})^{\star} = D^{\star}$, that is I is \star -invertible. By assumption, it follows that I is \star_f -invertible, and so I is \star_f -finite (Proposition 2.6). By Lemma 2.3, we conclude that there exists $J \in \mathbf{f}(D)$ with $J \subseteq I$ and $J^{\star} = I^{\star} = D^{\star}$. \square

Let \star be a semistar operation of D . If we denote by $\iota : D \hookrightarrow D^{\star}$ the embedding of D in D^{\star} and by \star_{ι} the (semi)star operation canonically induced on D^{\star} by \star (defined as in Proposition 1.3), we note that, if $I \in \text{Inv}(D, \star)$, then $I^{\star} \in \text{Inv}(D^{\star}, \star_{\iota})$. As a matter of fact, we have: $D^{\star} = (II^{-1})^{\star} = (I^{\star}(D : I)^{\star})^{\star} \subseteq (I^{\star}(D^{\star} : I^{\star}))^{\star} = (I^{\star}(D^{\star} : I^{\star}))^{\star_{\iota}} \subseteq (D^{\star})^{\star} = D^{\star}$.

Next example shows that the converse does not hold (in other words I^{\star} may be in $\text{Inv}(D^{\star}, \star_{\iota})$, with $I \in \mathbf{F}(D) \setminus \text{Inv}(D, \star)$), even if \star is a semistar operation stable and of finite type.

Example 2.9. Let D be an almost Dedekind domain, that is not a Dedekind domain (cf. for instance [23, Section 2 and the references]). Then, in D there exists a prime (= maximal) ideal P , such that P is not invertible (otherwise, D would be a Dedekind domain). Then, $P^{-1} = D$ [14, Corollary 3.1.3], since D is a Prüfer domain. Consider the semistar operation $\star := \star_{\{P\}}$. Let $\iota_P : D \hookrightarrow D_P$ be the canonical embedding, then $P^{\star} = PD_P$ is \star_{ι_P} -invertible, since D_P is a DVR, but $(PP^{-1})^{\star} = (PD)^{\star} = P^{\star} = PD_P \subsetneq D_P = D^{\star}$, thus P is not \star -invertible.

Let $\iota : D \hookrightarrow D^{\star}$ be the canonical embedding, then, we say that $I \in \overline{\mathbf{F}}(D)$ is *quasi- \star -invertible* if $I^{\star} \in \text{Inv}(D^{\star}, \star_{\iota})$ (that is, if $(I(D^{\star} : I))^{\star} = D^{\star}$). Note that

$I^* \in \text{Inv}(D^*, \star_\iota)$ implies that $I^* \in \mathbf{F}(D^*)$. We denote by $\text{QInv}(D, \star)$ the set of all quasi- \star -invertible D -submodules of K and, when $\star = d$, we set $\text{QInv}(D)$, instead of $\text{QInv}(D, d)$. We have already noticed that $\text{Inv}(D, \star) \subseteq \text{QInv}(D, \star)$ and that the inclusion can be proper. Moreover, it is obvious that $\text{QInv}(D) = \text{Inv}(D)$.

We have the following straightforward necessary and sufficient condition for a D -submodule of K to be quasi- \star -invertible.

Lemma 2.10. *Let \star be a semistar operation on an integral domain D and $I \in \overline{\mathbf{F}}(D)$. Then, I is quasi- \star -invertible if and only if there exists $H \in \overline{\mathbf{F}}(D)$ such that $(IH)^* = D^*$. \square*

Next we prove an analogue of Lemma 2.1 for quasi- \star -invertible ideals.

Lemma 2.11. *Let \star, \star_1, \star_2 be semistar operations on an integral domain D . Then:*

- (0) $D^* \in \text{QInv}(D, \star)$.
- (1) *If $\star_1 \leq \star_2$, then $\text{QInv}(D, \star_1) \subseteq \text{QInv}(D, \star_2)$. In particular, we have $\text{QInv}(D) \subseteq \text{QInv}(D, \tilde{\star}) \subseteq \text{QInv}(D, \star_f) \subseteq \text{QInv}(D, \star)$.*
- (2) $I, J \in \text{QInv}(D, \star)$ if and only if $IJ \in \text{QInv}(D, \star)$.
- (3) *If $I \in \text{QInv}(D, \star)$, then $(D^* : I) \in \text{QInv}(D, \star)$.*
- (4) *If $I \in \text{QInv}(D, \star)$, then $I^{v(D^*)} := (D^* : (D^* : I)) \in \text{QInv}(D, \star)$.*

Proof. (0) and (1) are straightforward. To prove (2) we notice that $I, J \in \text{QInv}(D, \star)$ if and only if $I^*, J^* \in \text{Inv}(D^*, \star_\iota)$, where \star_ι is defined as above. It follows (from Lemma 2.1(2)) that $I, J \in \text{QInv}(D, \star)$ if and only if $I^*J^* \in \text{Inv}(D^*, \star_\iota)$. It is easy to see that this happens if and only if $(IJ)^* \in \text{Inv}(D^*, \star_\iota)$, that is, if and only if $IJ \in \text{QInv}(D, \star)$. (3) is clear and (4) is an immediate consequence of Lemma 2.1(4) and of the fact that $(v(D^*))_\iota = v_{D^*}$, where v_{D^*} is the v -operation of D^* , ι is the canonical embedding of D in D^* and $v(D^*)$ is the semistar operation on D , defined by $E^{v(D^*)} := (D^* : (D^* : E))$, for each $E \in \overline{\mathbf{F}}(D)$ (note that, obviously, $\star \leq v(D^*)$). \square

Corollary 2.12. *Let \star be a semistar operation on an integral domain D , let $v(D^*)$ be the semistar operation on D , defined in the proof of Lemma 2.11(4) and let $I \in \overline{\mathbf{F}}(D)$. If I is quasi- \star -invertible, then I is quasi- $v(D^*)$ -invertible and $I^* = I^{v(D^*)}$.*

Proof. Let ι be the canonical embedding of D in D^* . As we noted in the proof of Lemma 2.11 (4), $(v(D^*))_\iota = v_{D^*}$. Then, in order to show that I^* is quasi- $v(D^*)$ -invertible, we prove that I^* is v_{D^*} -invertible. But \star_ι is a (semi)star operation on D^* and I^* is \star_ι -invertible, then (Lemma 2.1 (1)) I^* is v_{D^*} -invertible, since $\star_\iota \leq v_{D^*}$ [24, Theorem 34.1(4)]. Therefore I is quasi- $v(D^*)$ -invertible and $I^* = (I^{v(D^*)})^* = I^{v(D^*)}$, since $(D^* : I) = (D^* : I^{v(D^*)})$ (cf. also [51, p. 433] or [11, Lemma 2.1(3)], and Remark 2.13(b1)). \square

Remark 2.13. (a) Note that if I is a quasi- \star -invertible ideal of D , then every ideal J of D , with $I \subseteq J \subseteq I^* \cap D$, is also quasi- \star -invertible.

More precisely, let $I, J \in \mathbf{F}(D)$ [respectively, $I, J \in \overline{\mathbf{F}}(D)$], assume that $J \subseteq I$, $J^* = I^*$ and that I is \star -invertible [respectively, quasi- \star -invertible] then J is \star -invertible [respectively, quasi- \star -invertible].

Conversely, let $I, J \in \overline{\mathbf{F}}(D)$, assume that $J \subseteq I$, $J^* = I^*$ and that J is quasi- \star -invertible then I is quasi- \star -invertible (but not necessarily \star -invertible, even if J is \star -invertible).

As a matter of fact, if I is \star -invertible, then $D^* = (I(D : I))^* = (J(D : I))^* \subseteq (J(D : J))^* \subseteq D^*$. The quasi- \star -invertible case is similar. Conversely, if J is quasi- \star -invertible then $D^* = (J(D^* : J))^* = (I(D^* : J))^*$, thus I is quasi- \star -invertible and $(D^* : J) = (D^* : J)^* = (D^* : I)^* = (D^* : I)$ (cf. also (d1)).

Example 2.9 shows the parenthetical part of the statement. Let D , P and \star be as in Example 2.9. Note that P^\star is principal in (the DVR) $D^\star = D_P$, thus $P^\star = PD_P = tD_P$, for some nonzero $t \in PD_P$. Therefore, if $J := tD$, then $J^\star = P^\star$, i.e. P is \star -finite. We already observed that P is quasi- \star -invertible but not \star -invertible, even if obviously J is (\star) -invertible.

(b) Let $I, H', H'', J, L \in \overline{\mathbf{F}}(D)$. The following properties are straightforward:

$$(b1) \quad (IH')^\star = D^\star = (IH'')^\star \Rightarrow H'^\star = H''^\star = (D^\star : I)^\star = (D^\star : I).$$

$$(b2) \quad I \in \text{QInv}(D, \star), \quad IJ \subseteq IL \Rightarrow J^\star \subseteq L^\star.$$

$$(b3) \quad I \in \text{QInv}(D, \star), \quad J \subseteq I^\star \Rightarrow \exists L \in \overline{\mathbf{F}}(D), \quad (IL)^\star = J^\star.$$

[Take $L := (D^\star : I)J$.]

$$(b4) \quad I, J \in \text{QInv}(D, \star), \quad (IL)^\star = J^\star \Rightarrow L \in \text{QInv}(D, \star).$$

[Set $H := I(D^\star : J)$, and note that $(LH)^\star = D^\star$.]

$$(b5) \quad I, J \in \text{QInv}(D, \star) \Rightarrow (D^\star : IJ) = (D^\star : IJ)^\star = ((D^\star : I)(D^\star : J))^\star.$$

$$(b6) \quad I, J \in \text{QInv}(D, \star) \Rightarrow \exists L \in \text{QInv}(D, \star), \quad L \subseteq I^\star, \quad L \subseteq J^\star.$$

[Take $z \in K, z \neq 0$, such that $zI \subseteq D^\star, zJ \subseteq D^\star$, and set $L := zIJ$.]

$$(b7) \quad I, J \in \text{QInv}(D, \star), \quad I + J \in \text{QInv}(D, \star) \Rightarrow I^{v(D^\star)} \cap J^{v(D^\star)} \in \text{QInv}(D, \star).$$

[Recall that $\star \leq v(D^\star)$ and note that:

$$\begin{aligned} ((D^\star : I)(D^\star : J)(I + J))^\star &= (((D^\star : I)I)^\star (D^\star : J) + (D^\star : I)((D^\star : J)J)^\star)^\star \\ &= ((D^\star : J) + (D^\star : I))^\star = ((D^\star : J^{v(D^\star)}) + (D^\star : I^{v(D^\star)}))^\star \Rightarrow \\ &((D^\star : I)(D^\star : J)(I + J))^{v(D^\star)} = ((D^\star : I^{v(D^\star)}) + (D^\star : J^{v(D^\star)}))^{v(D^\star)} \Rightarrow \\ &(D^\star : ((D^\star : I)(D^\star : J)(I + J))) = (D^\star : ((D^\star : I^{v(D^\star)}) + (D^\star : J^{v(D^\star)}))) \\ &= (D^\star : (D^\star : I^{v(D^\star)})) \cap (D^\star : (D^\star : J^{v(D^\star)})) = I^{v(D^\star)} \cap J^{v(D^\star)}. \end{aligned}$$

$$(b8) \quad I, J \in \text{QInv}(D, \star), \quad I^{v(D^\star)} \cap J^{v(D^\star)} \in \text{QInv}(D, \star) \Rightarrow I + J \in \text{QInv}(D, v(D^\star)).$$

[Since $I^{v(D^\star)} \cap J^{v(D^\star)} = (D^\star : ((D^\star : I)(D^\star : J)(I + J)))$ and hence

$$(D^\star : (I^{v(D^\star)} \cap J^{v(D^\star)})) = ((D^\star : I)(D^\star : J)(I + J))^{v(D^\star)}, \text{ then apply (b4) to conclude that } I + J \in \text{QInv}(D, v(D^\star)).]$$

(c) A statement analogous to Corollary 2.12 holds for \star -invertibles: *Let \star be semistar operation on an integral domain D , let $v(D^\star)$ be the semistar operation on D , defined in the proof of Lemma 2.11(4) and let $I \in \mathbf{F}(D)$. If I is \star -invertible, then I is $v(D^\star)$ -invertible and $I^\star = I^{v(D^\star)}$.*

(d) *Mutatis mutandis*, the statements in (b) hold for \star -invertibles. More precisely: Let \star be a semistar operation on an integral domain D and let $I, H', H'', J, L \in \mathbf{F}(D)$, then:

$$(d1) \quad I \in \text{Inv}(D, \star), \quad (IH')^\star = D^\star = (IH'')^\star \Rightarrow H'^\star = H''^\star = (I^{-1})^\star.$$

$$(d2) \quad I \in \text{Inv}(D, \star), \quad IJ \subseteq IL \Rightarrow J^\star \subseteq L^\star.$$

$$(d3) \quad I \in \text{Inv}(D, \star), \quad J \subseteq I^\star \Rightarrow \exists L \in \mathbf{F}(D), \quad (IL)^\star = J^\star.$$

$$(d4) \quad I, J \in \text{Inv}(D, \star), \quad (IL)^\star = J^\star \Rightarrow L \in \text{QInv}(D, \star), \quad (D^\star : L) = (I(D : J))^\star.$$

Note that, under the present hypotheses, $L \in \text{Inv}(D, \star)$ if and only if $(D : L)^\star = (I(D : J))^\star$.

$$(d5) \quad I, J \in \text{Inv}(D, \star) \Rightarrow (D : IJ)^\star = ((D : I)(D : J))^\star.$$

$$(d6) \quad I, J \in \text{Inv}(D, \star) \Rightarrow \exists L \in \text{Inv}(D, \star), \quad L \subseteq I, \quad L \subseteq J.$$

$$(d7) \quad I, J \in \text{Inv}(D, \star), \quad I + J \in \text{Inv}(D, \star) \Rightarrow I^{v(D^\star)} \cap J^{v(D^\star)} \in \text{Inv}(D, \star).$$

$$(d8) \quad I, J \in \text{Inv}(D, \star), \quad I^{v(D^\star)} \cap J^{v(D^\star)} \in \text{Inv}(D, \star) \Rightarrow I + J \in \text{Inv}(D, v(D^\star)).$$

Our next goal is to extend Proposition 2.6 to the case of quasi- \star_f -invertibles. We need the following:

Lemma 2.14. *Let \star be a semistar operation on an integral domain D with quotient field K , let $\iota : D \hookrightarrow D^\star$ the embedding of D in D^\star , let \star_ι denote the (semi)star operation canonically induced on D^\star by \star and let $I \in \overline{\mathbf{F}}(D)$. Then, I is \star -finite if and only if I^\star is \star_ι -finite.*

Proof. If I is \star -finite, then there exists $J \in \mathbf{f}(D)$ such that $I^\star = J^\star$. It is clear that $(JD^\star)^{\star_\iota} = I^\star$, with $JD^\star \in \mathbf{f}(D^\star)$. Thus, I^\star is \star_ι -finite. Conversely, let I^\star be \star_ι -finite. Then, there exists $J_0 \in \mathbf{f}(D^\star)$, $J_0 = (a_1, a_2, \dots, a_n)D^\star$, with $\{a_1, a_2, \dots, a_n\} \subseteq K$, such that $J_0^\star = J_0^{\star_\iota} = I^{\star_\iota} = I^\star$. Set $J = (a_1, a_2, \dots, a_n)D \in \mathbf{f}(D)$. Then, $J^\star = (a_1D + a_2D + \dots + a_nD)^\star = (a_1D^\star + a_2D^\star + \dots + a_nD^\star)^\star = J_0^\star = I^\star$, and so I is \star -finite. \square

Proposition 2.15. *Let \star be a semistar operation on an integral domain D and let $I \in \overline{\mathbf{F}}(D)$. Then I is quasi- \star_f -invertible if and only if I and $(D^\star : I)$ are \star_f -finite (hence, \star -finite) and I is quasi- \star -invertible.*

Proof. Let $\iota : D \hookrightarrow D^\star$ be the canonical embedding and let \star_ι be the (semi)star operation on D^\star canonically induced by \star .

For the “if” part, use the same argument of the proof of the “if” part of Proposition 2.6.

The “only if” part. Since I is quasi- \star_f -invertible, then $(D^\star : I)$ is also quasi- \star_f -invertible, thus we have that I^{\star_f} and $(D^\star : I)^{\star_f} = (D^\star : I)$ are $(\star_f)_\iota$ -invertibles. Then, I^{\star_f} and $(D^\star : I)$ are $(\star_f)_\iota$ -finite (Corollary 2.6) and then I and $(D^\star : I)$ are \star_f -finite, by Lemma 2.14. Clearly I is quasi- \star -invertible, since $\star_f \leq \star$ (Lemma 2.11 (1)). \square

It is natural to ask under which conditions a quasi- \star -invertible fractional ideal is \star -invertible. Let $I \in \mathbf{F}(D)$ be quasi- \star -invertible. Then $(I(D^\star : I))^\star = D^\star$. Suppose that I is also \star -invertible, that is, $(I(D : I))^\star = D^\star$. Then, $(D : I)^\star = ((D : I)(I(D^\star : I))^\star)^\star = (((D : I)I)^\star (D^\star : I))^\star = (D^\star : I)^\star = (D^\star : I) = (D^\star : I^\star) \supseteq (D : I)^\star$. Therefore we have the following (cf. also Remark 2.2(b)):

Proposition 2.16. *Let \star be a semistar operation on an integral domain D . Let I be a quasi- \star -invertible fractional ideal of D . Then, I is \star -invertible if and only if $(D : I)^\star = (D^\star : I)$ (i.e. $(I^{-1})^\star = (I^\star)^{-1}$). \square*

The following corollary is straightforward (in particular, part (2) follows immediately from [13, proof of Remark 1.7]):

Corollary 2.17. *Let \star be a semistar operation on an integral domain D , and let $I \in \mathbf{F}(D)$.*

- (1) *If \star is a (semi)star operation then I is quasi- \star -invertible if and only if I is \star -invertible.*
- (2) *If \star is stable and $I \in \mathbf{f}(D)$ then I is quasi- \star -invertible if and only if I is \star -invertible. \square*

We notice that if \star is a semistar operation of finite type, \star -invertibility depends only on the set of quasi- \star -maximal ideals of D . Indeed, it is clear that $I \in \mathbf{F}(D)$ is \star -invertible if and only if $(II^{-1})^\star \cap D$ is not contained in any quasi- \star -maximal ideal. Then, from Lemma 1.2, we deduce immediately the following general result (cf. [13, Proposition 4.25]):

Proposition 2.18. *Let \star be a semistar operation on an integral domain D . Let $I \in \mathbf{F}(D)$. Then I is \star_f -invertible if and only if I is $\tilde{\star}$ -invertible. \square*

A classical example due to Heinzer can be used for describing the content of the previous proposition.

Example 2.19. Let K be a field and X an indeterminate over K . Set $D := K[[X^3, X^4, X^5]]$ and $M := (X^3, X^4, X^5)D$. It is easy to see that D is a one-dimensional Noetherian local integral domain, with maximal ideal M . Let $\star := v$, note that in this case $v = \star = \star_f = t$ and $\mathcal{M}(v) = \{M\}$, since $M = (D : K[[X]])$.

Therefore, $w = \tilde{v} = d$. In this situation $\text{Inv}(D, v) = \text{Inv}(D, t) = \text{Inv}(D, w) = \text{Inv}(D) = \{zD \mid z \in K, z \neq 0\}$. But $v = t \neq w = d$, because in general $(I \cap J)^t$ is different from $I^t \cap J^t$ in D , since D is not a Gorenstein domain [2, Theorem 5, Corollary 5.1] and [35, Theorem 222].

A result “analogous” to Proposition 2.18 does not hold, in general, for quasi-semistar-invertibility, as we show in the following:

Example 2.20. Let D be a pseudo-valuation domain, with maximal ideal M , such that $V := M^{-1}$ is a DVR (for instance, take two fields $k \subsetneq K$ and let $V := K[[X]]$, $M := XK[[X]]$ and $D := k + M$). Consider the semistar operation of finite type $\star := \star_{\{V\}}$, defined by $E^{\star_{\{V\}}} := EV$, for each $E \in \overline{\mathbf{F}}(D)$. It is clear that M is the only quasi- \star -maximal ideal of D . Thus, $\tilde{\star} = \star_{\{M\}} = d$, the identity (semi)star operation of D . We have $(M(V : M))^{\star} = (M(V : M))V = V$, since V is a DVR. Hence, M is quasi- \star -invertible. On the other side, M is not invertible (i.e., not quasi- $\tilde{\star}$ -invertible), since $MM^{-1} = MV = M$.

Under the assumption $D^{\star} = D^{\tilde{\star}}$ we have the following extension of Proposition 2.18 to the case of quasi-semistar-invertibility:

Proposition 2.21. *Let \star be a semistar operation on an integral domain D . Suppose that $D^{\star} = D^{\tilde{\star}}$. Let $I \in \overline{\mathbf{F}}(D)$. Then I is quasi- \star_f -invertible if and only if I is quasi- $\tilde{\star}$ -invertible*

Proof. If I is quasi- $\tilde{\star}$ -invertible, then there exists $J \in \overline{\mathbf{F}}(D)$ with $(IJ)^{\tilde{\star}} = D^{\tilde{\star}}$. This implies $(IJ)^{\star_f} = D^{\star_f}$, since $\tilde{\star} \leq \star_f$. Conversely, suppose that there exists $J \in \overline{\mathbf{F}}(D)$ such that $(IJ)^{\star_f} = D^{\star_f}$. Then $IJ \subseteq D^{\star_f} = D^{\star} = D^{\tilde{\star}}$. Thus, $(IJ)^{\tilde{\star}} \subseteq D^{\tilde{\star}}$. If $(IJ)^{\tilde{\star}} \subsetneq D^{\tilde{\star}}$, then $(IJ)^{\tilde{\star}} \cap D \subsetneq D$ is a quasi- $\tilde{\star}$ -ideal of D . It follows that $(IJ)^{\tilde{\star}} \cap D$ is contained in a quasi- $\tilde{\star}$ -maximal P of D . From Lemma 1.2, P is also a quasi- \star_f -maximal. Then, $(IJ)^{\star_f} \cap D \subseteq ((IJ)^{\tilde{\star}} \cap D)^{\star_f} \subseteq P^{\star_f} \subsetneq D^{\star_f}$, a contradiction. Then, I is quasi- $\tilde{\star}$ -invertible. \square

Remark 2.22. (a) If \star is a semistar operation on an integral domain D , we already observed (Remark 2.2(a)) that $\text{Inv}(D, \star)$ is not a group with respect to the standard multiplication of fractional ideals. In the set of the \star -invertible \star -fractional ideals, i.e. in the set $\text{Inv}^{\star}(D) := \{I \in \text{Inv}(D, \star) \mid I = I^{\star}\}$, we can introduce a *semistar composition* “ \times ” in the following way $I \times J := (IJ)^{\star}$. Note that $(\text{Inv}^{\star}(D), \times)$ is still not a group, in general, because for instance it does not possede an identity element (e.g. when $D^{\star} \in \overline{\mathbf{F}}(D) \setminus \mathbf{F}(D)$).

On the other hand, $\text{QInv}^{\star}(D) := \{I \in \text{QInv}(D, \star) \mid I = I^{\star}\}$, with the semistar composition “ \times ” introduced above, is always a group, having as identity D^{\star} and unique inverse of $I \in \text{QInv}^{\star}(D)$ the D -module $(D^{\star} : I) \in \overline{\mathbf{F}}(D)$, which belongs to $\text{QInv}^{\star}(D)$. This fact provides also one of the motivations for considering $\text{QInv}(D, \star)$ and $\text{QInv}^{\star}(D)$ (and not only $\text{Inv}(D, \star)$ and $\text{Inv}^{\star}(D)$, as in the “classical” star case).

It is not difficult to prove that: *let \star be a semistar operation on an integral domain D , then:*

$$(\text{Inv}^{\star}(D), \times) \text{ is a group} \Leftrightarrow (D : D^{\star}) \neq (0).$$

As a matter of fact, (\Rightarrow) holds because $D^{\star} \in \text{Inv}^{\star}(D) \subseteq \mathbf{F}(D)$ and so $(D : D^{\star}) \neq (0)$. (\Leftarrow) holds because $(D : D^{\star}) \neq (0)$ implies that $D^{\star} \in \text{Inv}^{\star}(D)$ and, for each $I \in \text{Inv}^{\star}(D)$, we have $(D^{\star} : I) \in \mathbf{F}(D)$, thus $(D : I)^{\star} = (D^{\star} : I)$ (Remark 2.13(d1)) and so the inverse of each element $I \in \text{Inv}^{\star}(D)$ exists and is uniquely determined in $\text{Inv}^{\star}(D)$.

Note that, even if $(\text{Inv}^{\star}(D), \times)$ is a group, $\text{Inv}^{\star}(D)$ could be a proper subset of $\text{QInv}^{\star}(D)$. For this purpose, take D , V , M as in Example 2.20, in this case

$D^\star = V$ and $(D : V) = M \neq (0)$, hence $(\text{Inv}^\star(D), \times)$ is a group, but $M \in \text{QInv}^\star(D) \setminus \text{Inv}^\star(D)$.

(b) Note that, if \star is a semistar operation on an integral domain D , the group $(\text{QInv}^\star(D), \times)$ can be identified with a more classic group of star-invertible star-ideals. As a matter of fact, it is easy to see that:

$$(\text{QInv}^\star(D), \times) = (\text{Inv}^{\star_\iota}(D^\star), \times')$$

where $\iota : D \rightarrow D^\star$ is the canonical embedding, \star_ι is the (semi)star operation on D^\star , canonically associated to \star (Proposition 1.3), and the (semi)star composition “ \times' ” in $\text{Inv}^{\star_\iota}(D^\star)$ is defined by $E \times' F := (EF)^{\star_\iota}$.

(c) Let \star_1, \star_2 be two semistar operations on an integral domain D . If $\star_1 \leq \star_2$ then $\text{Inv}(D, \star_1) \subseteq \text{Inv}(D, \star_2)$ and $\text{QInv}(D, \star_1) \subseteq \text{QInv}(D, \star_2)$. Note that it is not true in general that $\text{Inv}^{\star_1}(D) \subseteq \text{Inv}^{\star_2}(D)$ or that $\text{QInv}^{\star_1}(D) \subseteq \text{QInv}^{\star_2}(D)$, because there is no reason for a \star_1 -ideal (or \star_1 -module) to be a \star_2 -ideal (or \star_2 -module). For instance, let T be a proper overring of an integral domain D , let $\star_1 := d$ be the identity (semi)star operation on D and let $\star_1 := \star_{\{T\}}$ be the semistar operation on D defined by $E^{\star_{\{T\}}} := ET$, for each $E \in \overline{\mathbf{F}}(D)$. If I is a nonzero principal ideal of D , then obviously $I \in \text{Inv}^{\star_1}(D)$ ($= \text{Inv}(D) = \text{QInv}^{\star_1}(D)$) but I does not belong to $\text{QInv}^{\star_2}(D)$ (and, in particular, it does not belong to $\text{Inv}^{\star_2}(D)$), because $I^{\star_2} = IT \neq I$.

Note that, even if $\text{Inv}(D, \star_1) = \text{Inv}(D, \star_2)$, for some pair of semistar operations $\star_1 \leq \star_2$, it is not true in general that $\text{Inv}^{\star_1}(D) \subseteq \text{Inv}^{\star_2}(D)$. Take D, V, M as in Example 2.20. Let $\star_1 := d$ be the identity (semi)star operations on D and let $\star_2 := \star_{\{V\}}$. In this case, $\text{Inv}(D, \star_1) = \text{Inv}(D, \star_2)$, because $\star_1 = \widetilde{\star_2}$ and $\star_2 = (\star_2)_f$ (Proposition 2.18). But, $\text{Inv}^{\star_2}(D) \subsetneq \text{Inv}^{\star_1}(D) = \text{Inv}(D)$, because $\text{Inv}^{\star_2}(D) \subseteq \text{Inv}^{\star_1}(D) = \text{Inv}(D)$ since each \star_2 -ideal is obviously a \star_1 -ideal, and moreover the proper inclusion holds because, as above, a nonzero principal ideal of D belongs to $\text{Inv}(D)$ but not to $\text{Inv}^{\star_2}(D)$.

On the other hand, if $\star_1 \leq \star_2$ are two *star operations* on D , then it is known that $\text{Inv}^{\star_1}(D) \subseteq \text{Inv}^{\star_2}(D)$, essentially because, in this case, $I \in \text{Inv}^{\star_1}(D)$ implies that $I = I^{\star_1} = I^v$ and so $I = I^{\star_2}$ [6, Proposition 3.3].

(d) Let \star be a semistar operation on an integral domain D , let $v(D^\star)$ be the semistar operation on D defined in Lemma 2.11(4) and let $I, J \in \mathbf{F}(D)$ [respectively, $I, J \in \overline{\mathbf{F}}(D)$]. Assume that I is a \star -invertible [respectively, quasi- \star -invertible] \star -ideal of D , then:

$$(IJ^v)^\star = (I : (D : J)) \text{ [respectively, } (IJ^{v(D^\star)})^\star = (I : (D^\star : J))].$$

Recall that, since $I = I^\star$, then $(I : (D : J)) = (I : (D : J))^\star$. It is obvious that $IJ^v \subseteq (I : (D : J^v)) = (I : (D : J))$ and thus $(IJ^v)^\star \subseteq (I : (D : J))$. Conversely, if $z \in (I : (D : J))$ then $z(D : J) \subseteq I$ and so $z(D : I) \subseteq J^v$. Therefore $z \in zD^\star = z((D : I)I)^\star \subseteq (IJ^v)^\star$.

For the quasi- \star -invertible case, if $I = I^\star$, then $(I : (D^\star : J)) = (I : (D^\star : J))^\star$ and $I = ID^\star$. It is obvious that $IJ^{v(D^\star)} \subseteq (I : (D^\star : J^{v(D^\star)})) = (I : (D^\star : J))$ and thus $(IJ^{v(D^\star)})^\star \subseteq (I : (D^\star : J))$. Conversely, if $z \in (I : (D^\star : J))$ then $z(D^\star : J) \subseteq I$ and so $z(D^\star : I) \subseteq J^{v(D^\star)}$. Therefore $z \in zD^\star = z((D^\star : I)I)^\star \subseteq (IJ^{v(D^\star)})^\star$.

In the next theorem, we investigate the behaviour of a \star -invertible ideal (when \star is a semistar operation) with respect to the localizations at quasi- \star -maximal ideals and in the passage to semistar Nagata ring. More precisely, in the spirit of Kaplansky's theorem on $(d-)$ invertibility [35, Theorem 62], we extend a characterization of t -invertibility proved in [36, Corollary 3.2] and two Kang's results proved in the star setting [34, Theorem 2.4 and Proposition 2.6].

Theorem 2.23. *Let \star be a semistar operation on an integral domain D . Assume that $\star = \star_f$. Let $I \in \mathbf{f}(D)$, then the following are equivalent:*

- (i) I is \star -invertible.
- (ii) $ID_Q \in \text{Inv}(D_Q)$, for each $Q \in \mathcal{M}(\star)$ (and then ID_Q is principal in D_Q).
- (iii) $I\text{Na}(D, \star) \in \text{Inv}(\text{Na}(D, \star))$.

Proof. (i) \Rightarrow (ii). If $(II^{-1})^\star = D^\star$, then $II^{-1} \not\subseteq Q$, for each $Q \in \mathcal{M}(\star)$. Since $I \in \mathbf{f}(D)$, by flatness we have:

$$I^{-1}D_Q = (D : I)D_Q = (D_Q : ID_Q) = (ID_Q)^{-1}.$$

Therefore, for each $Q \in \mathcal{M}(\star)$, since $II^{-1} \not\subseteq Q$, we have:

$$D_Q = (II^{-1})D_Q = ID_Q I^{-1}D_Q = ID_Q(ID_Q)^{-1}.$$

(ii) \Rightarrow (iii). From the assumption and from the proof of (i) \Rightarrow (ii), we have that $II^{-1} \not\subseteq Q$, for each $Q \in \mathcal{M}(\star)$. Since $I \in \mathbf{f}(D)$, by the flatness of the canonical homomorphism $D \rightarrow D[X]_{N(\star)} = \text{Na}(D, \star)$, we have:

$$(I[X]_{N(\star)})^{-1} = (D[X]_{N(\star)} : I[X]_{N(\star)}) = (D : I)[X]_{N(\star)} = I^{-1}[X]_{N(\star)}.$$

Since $II^{-1} \not\subseteq Q$, then $(II^{-1})[X]_{N(\star)} \not\subseteq Q[X]_{N(\star)}$, for each $Q \in \mathcal{M}(\star)$. From [19, Proposition 3.1(3)], we deduce that:

$$D[X]_{N(\star)} = (II^{-1})[X]_{N(\star)} = I[X]_{N(\star)}(I[X]_{N(\star)})^{-1},$$

where $I\text{Na}(D, \star) = I[X]_{N(\star)}$.

(iii) \Rightarrow (i). From the assumption and from the previous considerations, we have:

$$D[X]_{N(\star)} = I[X]_{N(\star)}(I[X]_{N(\star)})^{-1} = (II^{-1})[X]_{N(\star)},$$

and thus $(II^{-1})[X]_{N(\star)} \not\subseteq Q[X]_{N(\star)}$, for each $Q \in \mathcal{M}(\star)$. This fact implies that $II^{-1} \not\subseteq Q$, for each $Q \in \mathcal{M}(\star)$. From [19, Lemma 2.4 (1)], we deduce immediately that $(II^{-1})^\star = D^\star$. \square

Corollary 2.24. *Let \star be a stable semistar operation of finite type on D , and let $I \in \mathbf{f}(D)$. Then, the conditions (i)–(iii) of Theorem 2.23 are equivalent to:*

- (iv) I is quasi- \star -invertible.

Proof. Apply Corollary 2.17. \square

Remark 2.25. It is known [34, Proposition 2.6] (cf. also [5, Section 1] and [11, Section 1]) that, if \star is a star operation of finite type on an integral domain D , an ideal I of D is \star -invertible if and only if it is \star -finite and locally principal (when localized at the \star -maximal ideals). As a matter of fact, by Corollary 2.6, we have that, if I is \star -invertible, then I is \star -finite. Moreover, $(II^{-1})^\star = D$ implies $II^{-1} \not\subseteq Q$, for each \star -maximal ideal Q of D . It follows that $ID_Q I^{-1}D_Q = D_Q$. Thus, ID_Q is invertible (hence, principal) in D_Q . Conversely, assume that $I^\star = J^\star$, with $J \in \mathbf{f}(D)$, $J \subseteq I$. It is clear that $I^{-1} = J^{-1}$, since $I^v = (I^\star)^v = (J^\star)^v = J^v$, being $\star \leq v$ [24, Theorem 34.1(4)]. Suppose that I is not \star -invertible, that is, $(II^{-1})^\star \subsetneq D$. Then, there exists a \star -maximal ideal Q of D , such that $II^{-1} \subseteq Q$. It follows $QD_Q \supseteq ID_Q I^{-1}D_Q = ID_Q J^{-1}D_Q = ID_Q(JD_Q)^{-1} \supseteq ID_Q(ID_Q)^{-1}$, a contradiction, since ID_Q is principal.

We will see in a moment that the “if” part of a similar result for semistar operations does not hold, even if $I = I^\star$. More precisely, we can extend partially [21, Proposition 1.1] in the following way:

Let $I \in \mathbf{F}(D)$ and let \star be a semistar operation on D , the following properties are equivalent:

- (i) I is \star_f -invertible;
- (ii) $(Q : I) \subsetneq (D : I)$, for each $Q \in \mathcal{M}(\star_f)$;

(iii) $(Q : I) \subsetneq (D : I)$, for each $Q \in \mathcal{M}(\star_f)$ and $Q \supseteq I(D : I)$.

Moreover, each of the previous properties implies the following:

(iv) I is \star_f -finite and $ID_Q \in \text{Inv}(D_Q)$, for each $Q \in \mathcal{M}(\star_f)$ (and so ID_Q is principal in D_Q).

As a matter of fact, (i) \Rightarrow (ii) because $D^\star = (I(D : I))^\star$ and if $(Q : I) = (D : I)$, for some $Q \in \mathcal{M}(\star_f)$, then $I(D : I) = I(Q : I) \subseteq Q$, thus $(I(D : I))^\star \subseteq Q^{\star_f} \subsetneq D^\star$, hence we reach a contradiction. (ii) \Rightarrow (iii) is trivial. (iii) \Rightarrow (i): if not, $I(D : I) \subseteq Q$, for some $Q \in \mathcal{M}(\star_f)$, thus $(D : I) \subseteq (Q : I)$ and hence $(D : I) = (Q : I)$, which contradicts (iii).

Finally (ii) \Rightarrow (iv), because of Proposition 2.6 and because for $z_Q \in (D : I) \setminus (Q : I)$, we have $z_Q I \subset D \setminus Q$, and so $z_Q ID_Q = D_Q$, i.e. $ID_Q = (z_Q)^{-1} D_Q$, for each $Q \in \mathcal{M}(\star_f)$.

But note that, in the semistar setting, (iv) \nRightarrow (i), even in case I is a \star_f -ideal, \star_f -finite, as the following example will show. However, we can re-establish a characterization in the quasi- \star -invertibility setting in the following way: *if \star is a semistar operation of finite type on an integral domain D and if $I \in \overline{\mathbf{F}}(D)$, then $I \in \text{QInv}(D, \star)$ if and only if I^\star is \star -finite and $I^\star D_M^\star$ is principal, for each \star -maximal ideal M of D^\star .*

Example 2.26. Let D be a valuation domain, P a nonzero nonmaximal non-invertible prime ideal of D such that D_P is a discrete valuation domain. (For instance, if K is a field and X, Y are two indeterminates over K , let $D := K + XK[X]_{(X)} + YK(X)[Y]_{(Y)}$ and $P := YK(X)[Y]_{(Y)}$; in this case D is a two-dimensional valuation domain, $D_P = K(X)[Y]_{(Y)}$ and $P = PD_P = YD_P \subsetneq YD$.) Set $\star := \star_{\{P\}}$. In this situation, $\star = \star_f$ and $\mathcal{M}(\star) = \{P\}$, thus $\star = \widetilde{\star}$, i.e. \star is a stable semistar operation of finite type on D . Note that P is in fact a \star -ideal of D , since $P^\star = PD_P = P$. Moreover, $P^\star = PD_P = tD_P = (tD)^\star$ for some nonzero $t \in D_P$, i.e. P is a non zero principal ideal in $D^\star = D_P$, since D_P is a DVR, by assumption. Thus, P is a \star -ideal, \star -finite and locally principal, when localized at the quasi- \star -maximal ideal(s) of D . But P is not \star -invertible, since in this situation $(D : P) = (P : P) = D_P$ [14, Proposition 3.1.5] and hence $(P(D : P))^\star = (P(P : P))^\star = (PD_P)^\star = P^\star = P$. Note also that, in this situation, P is quasi- \star -invertible (because $(P(D^\star : P))^\star = (tD_P t^{-1} D_P)^\star = D_P = D^\star$) and $D^\star = D_P = (PD_P : PD_P) = (P : P)D_P = (P : P)^\star$.

Next two results generalize to the semistar setting [34, Theorem 2.12 and Theorem 2.14].

Corollary 2.27. *Let \star be a semistar operation on an integral domain D . Assume that $\star = \star_f$. Let $h \in D[X], h \neq 0$, then:*

$$\mathbf{c}(h) \in \text{Inv}(D, \star) \quad \Leftrightarrow \quad h \text{ Na}(D, \star) = \mathbf{c}(h) \text{ Na}(D, \star).$$

In particular, $\mathbf{c}(h) \in \text{Inv}(D, \star)$ if and only if $\mathbf{c}(h) \in \text{QInv}(D, \star)$.

Proof. The proof of the first part of the statement is based on the following result by D.D. Anderson [1, Theorem 1]: If R is a ring and $h \in R[X], h \neq 0$, then $hR(X) \subseteq \mathbf{c}(h)R(X)$ and, moreover, the following are equivalent:

- (1) $hR(X) = \mathbf{c}(h)R(X)$.
- (2) $\mathbf{c}(h)$ is locally principal (in R).
- (3) $\mathbf{c}(h)R(X)$ is principal (in $R(X)$).

(\Rightarrow) By Theorem 2.23 ((i) \Rightarrow (ii)), we have that $\mathbf{c}(h)D_Q$ is principal, for each $Q \in \mathcal{M}(\star)$. Hence,

$$\mathbf{c}(h)D_Q[X]_{N(\star)} = \mathbf{c}(h)(D[X]_{N(\star)})_{QD[X]_{N(\star)}} = \mathbf{c}(h)D_Q(X)$$

is principal, for each $Q \in \mathcal{M}(\star)$. By applying Anderson's result to the local ring $R = D_Q$, we deduce that $hD_Q(X) = c(h)D_Q(X)$, for each $Q \in \mathcal{M}(\star)$. The conclusion follows from Proposition 1.4, (2) and (3)

(\Leftarrow) If $h\text{Na}(D, \star) = c(h)\text{Na}(D, \star)$, then by localization we obtain that $hD_Q(X) = c(h)D_Q(X)$, for each $Q \in \mathcal{M}(\star)$ (Proposition 1.4 and [24, Corollary 5.3]). By Anderson's result, we deduce that $c(h)D_Q$ is principal, i.e. $c(h)D_Q \in \text{Inv}(D_Q)$, for each $Q \in \mathcal{M}(\star)$. The conclusion follows from Theorem 2.23 ((ii) \Rightarrow (i)).

The last part of the statement follows from the fact that $\text{Na}(D, \star) = \text{Na}(D, \widetilde{\star})$ [19, Corollary 3.5(3)] and from Corollary 2.17 and Proposition 2.18 or, directly, from Corollary 2.24. \square

Proposition 2.28. *Let \star be a semistar operation on an integral domain D . If H is an invertible ideal of $\text{Na}(D, \star)$, then H is principal in $\text{Na}(D, \star)$.*

Proof. We can assume that $H \in \text{Inv}(\text{Na}(D, \star))$ and $H \subseteq \text{Na}(D, \star)$, then, in particular, $H = (h_1, h_2, \dots, h_n)\text{Na}(D, \star)$, with $h_i \in D[X]$, $1 \leq i \leq n$. For each $Q \in \mathcal{M}(\star_f)$, by localization, we obtain that $HD_Q(X) = (h_1, h_2, \dots, h_n)D_Q(X)$ is a nonzero principal ideal (Theorem 2.23 ((iii) \Rightarrow (ii))). By a standard argument, if $d_i := \deg(h_i)$, for $1 \leq i \leq n$, and if

$$h := h_1 + h_2X^{d_1+1} + h_3X^{d_1+d_2+2} + \dots + h_nX^{d_1+d_2+\dots+d_{n-1}+n-1} \in D[X],$$

then it is not difficult to see that $HD_Q(X) = hD_Q(X)$, for each $Q \in \mathcal{M}(\star_f)$. From Proposition 1.4(3), we deduce that $H\text{Na}(D, \star) = h\text{Na}(D, \star)$. \square

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